Exercise 9

Solve the differential equation.

$$y'' - 4y' + 13y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} - 4(re^{rx}) + 13(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 4r + 13 = 0$$

Solve for r.

$$r = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

$$r = \{2 - 3i, 2 + 3i\}$$

Two solutions to the ODE are $e^{(2-3i)x}$ and $e^{(2+3i)x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{(2-3i)x} + C_2 e^{(2+3i)x} \\ &= C_1 e^{2x} e^{-3ix} + C_2 e^{2x} e^{3ix} \\ &= e^{2x}(C_1 e^{-3ix} + C_2 e^{3ix}) \\ &= e^{2x}[C_1(\cos 3x - i \sin 3x) + C_2(\cos 3x + i \sin 3x)] \\ &= e^{2x}[(C_1 + C_2)\cos 3x + (-iC_1 + iC_2)\sin 3x] \\ &= e^{2x}(C_3 \cos 3x + C_4 \sin 3x), \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.